Four-Spinor Reference Sheets

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Abstract

Some facts about 4-spinors listed and discussed. None, well perhaps some, of the work is original. However, locating formulas in other places has proved a time-consuming process in which one must always worry that the formulas found in any given source assume the other metric (I use $\{-1,-1,-1,+1\}$) or assume some other unexpected preconditions. Here I list some formulas valid in general representations first, then formulas using a chiral representation are displayed, and finally formulas in a special reference frame (the rest frame of the 'current' j) in the chiral representation are listed. Some numerical and algebraic exercises are provided.

1 General Representation

We can use any four complex numbers as the components of a 4-spinor in a given representation, $\psi = \text{col}\{a+bi, c+di, e+fi, g+hi\}$, where 'col' indicates a column matrix and the eight numbers $a...h$ are real. The 4-spinor generates four real-valued vectors: two light-like, one time-like and one space-like. These may be defined using the gamma matrices of the representation as follows:

$$j^\mu \equiv \overline{\psi}\gamma^\mu\psi \;\; ; \;\; a^\mu \equiv \overline{\psi}\gamma^\mu\gamma^5\psi \;\; ; \;\; r^\mu \equiv \overline{\psi}\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)\psi \;\; ; \;\; s^\mu \equiv \overline{\psi}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\psi, \quad (1)$$

where $\overline{\psi} \equiv \psi^\dagger\gamma^4$, μ is one of {1,2,3,4}, and $\gamma^5 \equiv -i\gamma^1\gamma^2\gamma^3\gamma^4$. Note that the vectors are representation independent; the substitution $\gamma^\mu \rightarrow S^{-1}\gamma^\mu S$ and $\psi \rightarrow S^{-1}\psi$ doesn't change the vectors. By using a specific representation, perhaps the one displayed below in (3), one can show after some algebra that (i) r and s are light-like vectors and that (ii) j is time-like and that (iii) a is space-like. An exception occurs (iv) when r or s is zero; then j and a are light-like.

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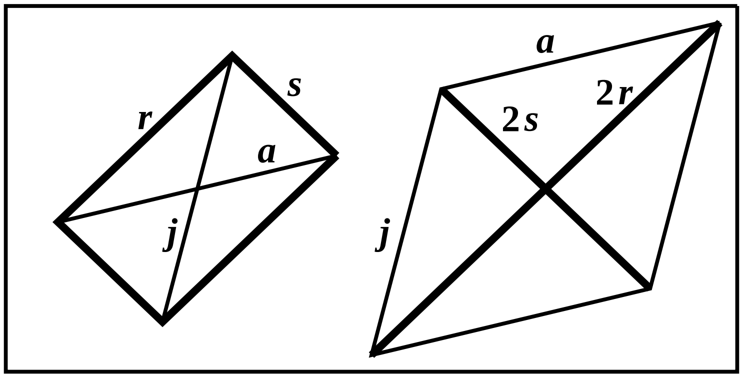

Figure 1: The vectors make parallelograms.

Since the gammas in (1) are sandwiched between common factors of $\overline{\psi}$ and ψ, we see that the following are true:

$$j^\mu = r^\mu + s^\mu \ ; \ a^\mu = r^\mu - s^\mu \ ; \ 2r^\mu = j^\mu + a^\mu \ ; \ 2s^\mu = j^\mu - a^\mu. \tag{2}$$

The vectors can be arranged in parallelograms, see Fig. 1.

The scalar product of j with itself, $j^2 \equiv j^\mu j_\mu$, is the same as that for a, $a^\mu a_\mu = -j^2$, except for the sign. The two vectors are 'orthogonal', $j^\mu a_\mu = 0$. We collect scalar products in Table 1.

Table 1: Scalar products.

Vector	j	a	r	s
j	j^2	0	$j^2/2$	$j^2/2$
a		$-j^2$	$-j^2/2$	$j^2/2$
r			0	$j^2/2$
s				0

2 Chiral Representation [CR]

To get specific formulas for the vectors in terms of the components of the 4-spinor ψ one must choose a representation for the gammas. I choose a chiral representation [CR]:

$$\gamma^k = \begin{pmatrix} 0 & +\sigma^k e^{i\delta} \\ -\sigma^k e^{-i\delta} & 0 \end{pmatrix}; \gamma^4 = \begin{pmatrix} 0 & -e^{i\delta} \\ -e^{-i\delta} & 0 \end{pmatrix}; \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{[CR]} \tag{3}$$

where δ is an arbitrary phase angle, k is any one of {1,2,3}, '1' is the unit 2x2 matrix, and the Pauli matrices are the 2x2 matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

One may check that the gammas (3) satisfy $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \cdot 1$, where '1' is the unit 4x4 matrix and $g^{\mu\nu} = \mathrm{diag}\{-1,-1,-1,+1\}$ is the 4x4 metric tensor.

Write the 4-spinor ψ as follows

$$\psi = \begin{pmatrix} r\cos(\theta_R/2)\exp(-\frac{i\phi_R}{2})\exp(i\frac{\alpha-\beta}{2}) \\ r\sin(\theta_R/2)\exp(+\frac{i\phi_R}{2})\exp(i\frac{\alpha-\beta}{2}) \\ l\cos(\theta_L/2)\exp(-\frac{i\phi_L}{2})\exp(i\frac{\alpha+\beta}{2}) \\ l\sin(\theta_L/2)\exp(+\frac{i\phi_L}{2})\exp(i\frac{\alpha+\beta}{2}) \end{pmatrix}. \quad \text{[CR]} \tag{5}$$

The given four complex numbers making up the components of ψ determine the eight real numbers r, θ_R, ϕ_R, l, θ_L, ϕ_L, α, and β, within the usual additive $n\pi$'s. By (1), (3), and (5) one finds an expression for j^2:

$$j^2 = 2r^2l^2(1 + \cos\theta_R\cos\theta_L + \cos\phi_R\cos\phi_L\sin\theta_R\sin\theta_L + \sin\phi_R\sin\phi_L\sin\theta_R\sin\theta_L). \tag{6}$$

[CR]

By (1), with the parameters in (5) and the representation (3), one finds specific formulas for r and s,

$$\{r^1, r^2, r^3, r^4\} = \{r^2\sin\theta_R\cos\phi_R, r^2\sin\theta_R\sin\phi_R, r^2\cos\theta_R, r^2\}; \quad \text{[CR]} \tag{7}$$

$$\{s^1, s^2, s^3, s^4\} = \{-l^2\sin\theta_L\cos\phi_L, -l^2\sin\theta_L\sin\phi_L, -l^2\cos\theta_L, l^2\}. \quad \text{[CR]} \tag{8}$$

Clearly the angles θ and ϕ are polar and azimuthal angles of the spatial directions of r and s. Specific formulas for j and a follow immediately from (2), (7), and (8).

With the chiral representation the 4-spinor splits into two 2-spinors, $\psi = \mathrm{col}\{\rho, \lambda\}$, where 'col' means column matrix. The 2-spinor ρ is right-handed and the other, λ, is left-handed,

referring to their Lorentz transformation properties. By (5), (7), and (8) one sees that the right 2-spinor ρ determines r and the left 2-spinor λ determines s. The 2x2 rotation matrix $R(\kappa, \hat{\mathbf{n}})$ for a rotation through an angle κ about the direction $\hat{\mathbf{n}}$ is the same for both right and left 2-spinors, $R(\kappa, \hat{\mathbf{n}}) = \exp(-i\hat{\mathbf{n}}_k \sigma^k \kappa/2)$. The 2x2 boost matrix $B(u, \hat{\mathbf{n}})$ for a boost of speed $\tanh u$ in the direction $\hat{\mathbf{n}}$ differs for right and left 2-spinors: $B_R(u, \hat{\mathbf{n}}) = \exp(+\hat{\mathbf{n}}_k \sigma^k u/2)$ and $B_L(u, \hat{\mathbf{n}}) = \exp(-\hat{\mathbf{n}}_k \sigma^k u/2)$.

A rotation through an angle κ about the direction $\hat{\mathbf{n}}$ changes the 4-spinor ψ: $\psi \rightarrow [\cos(\kappa/2) \cdot 1 - i\sin(\kappa/2) n_k \gamma^5 \gamma^4 \gamma^k]\psi$, where '1' is the unit 4x4 matrix. The rotation through κ about $\hat{\mathbf{n}} = \{0,0,1\}$ changes $\{j^1, j^2\}$ to $\{\cos\kappa j^1 - \sin\kappa j^2, \sin\kappa j^1 + \cos\kappa j^2\}$, leaving j^3 and j^4 unchanged.

A boost of speed $\tanh u$ in the direction $\hat{\mathbf{n}}$ changes the 4-spinor ψ: $\psi \rightarrow [\cosh(u/2) \cdot 1 + \sinh(u/2) n_k \gamma^4 \gamma^k]\psi$, where '1' is the unit 4x4 matrix. The boost of speed $\tanh u$ in the direction $\hat{\mathbf{n}} = \{0,0,1\}$ changes $\{j^3, j^4\}$ to $\{\cosh u j^3 + \sinh u j^4, \sinh u j^3 + \cosh u j^4\}$, leaving j^1 and j^2 unchanged.

3 j-time frame

By applying the appropriate boost (3 parameters: u, $\hat{n}^1$, $\hat{n}^2$ which determines $\hat{n}^3$) we get a new j which has no spatial components; the new j is in its proper frame. Call this the 'j-time frame.' In this frame the spinor has equal right and left 2-spinors within a phase, $\rho = e^{-i\beta}\lambda$, and the light-like vectors r and s point in opposite directions. The transformed 4-spinor may be written in the form

$$\psi = \sqrt{\frac{j}{2}} \begin{pmatrix} \cos(\theta/2)\exp(-\frac{i\phi}{2})\exp(i\frac{\alpha-\beta}{2}) \\ \sin(\theta/2)\exp(+\frac{i\phi}{2})\exp(i\frac{\alpha-\beta}{2}) \\ \cos(\theta/2)\exp(-\frac{i\phi}{2})\exp(i\frac{\alpha+\beta}{2}) \\ \sin(\theta/2)\exp(+\frac{i\phi}{2})\exp(i\frac{\alpha+\beta}{2}) \end{pmatrix}, \quad \text{[CR]} \tag{9}$$

(i) $[\{\theta, \phi\}]$ where $\{\theta, \phi\}$ are the { polar, azimuthal } angles indicating the direction of $\mathbf{r}$ and $\mathbf{a}$ which is opposite to the direction of $\mathbf{s}$. The overall phase is $\alpha/2$ and the phase shift from the right 2-spinor to the left 2-spinor is β. The four angles $\{\theta, \phi, \alpha, \beta\}$, the magnitude of j, and the three parameters u, $\hat{n}^1$, $\hat{n}^2$ of the boost amount to eight real numbers which is the same number needed to specify the four complex numbers making up a 4-spinor in a given representation. Thus we still have a general form for the 4-spinor.

(ii) $[\alpha]$ Rotating ψ in the j-time frame, (9), leaves j alone and changes the values of $\{\theta, \phi, \alpha\}$. If the rotation axis is in the direction of $\mathbf{a}$, $\hat{n}^k = a^k/\sqrt{j^2 + (a^4)^2}$ with $a^4 = 0$ in this frame, then the effect on α is especially simple: α changes by the negative of the rotation

angle κ, $\alpha \rightarrow \alpha - \kappa$. Rotating by $\kappa = \alpha$ about $\mathbf{a}$ brings α to zero, $\alpha \rightarrow 0$. Therefore we may interpret α, twice the overall phase of ψ in this frame, as a rotation angle.

The way this works can be seen as follows. When the direction $\mathbf{a}$ is along {1,0,0}, the angles θ and ϕ in (9) are $\theta = \pi/2$ and $\phi = 0$ or π. For $\phi = 0$ the right and left 2-spinors are given by $\rho = \lambda = \exp(i\alpha/2)$ col{1,1} if we take $\beta = 0$ and $j = 4$. As noted above, the effect of a rotation is to multiply both ρ and λ by the same 2x2 matrix $R(\kappa, \hat{\mathbf{n}})$. The rotation matrix $\exp(-i\sigma^1\kappa/2)$ for $\hat{\mathbf{n}} = \{1,0,0\}$ is a linear combination of the Pauli matrix σ^1 and the unit 2x2 matrix. But the 2-spinors are eigenspinors of σ^1 and the unit 2x2 matrix with eigenvalue 1, so the effect of the rotation matrix $\exp(-i\sigma^1\kappa/2)$ is to change the phase of ρ and λ by $-\kappa/2$. In short, the two 2-spinors are eigenspinors of the rotation matrix with the same eigenvalue which is the common phase factor $\exp(-i\kappa/2)$.

For $\phi = \pi$, the 2-spinor $\rho = \lambda = \exp(i\alpha/2)$ col{-1,1} is an eigenspinor of σ^1 with eigenvalue -1, so the common phase factor is $\exp(+i\kappa/2)$. In Table 2, we collect the change in angles $\{\theta, \phi, \alpha\}$ due to rotations of angle κ about the coordinate axes.

(iii) [β] The phase β is changed, $\beta \rightarrow \beta \pm \kappa$ sign depending on eigenvalue, when the right-handed 2-spinor ρ is rotated by κ and λ is rotated through $-\kappa$, both rotations taking place about $\mathbf{a}$. In this case none of the angles $\{\theta, \phi, \alpha\}$ changes and the magnitude of j doesn't change.

(iv) [j] An operation that changes only the magnitude of j while leaving $\{\theta, \phi, \alpha, \beta\}$ alone can be found. If the right 2-spinor ρ is boosted along the direction of $\mathbf{a}$ by $\tanh u$ and λ is boosted by the same speed but in the opposite direction $-\mathbf{a}$, then the magnitude of j changes, $j \rightarrow [\cosh u - \sinh u]j$.

Thus the 4-spinor parameters $\{\theta, \phi, \alpha\}$ can each be changed by a suitable rotation applied to ψ, β alone can be changed by applying a counter-clockwise rotation to the right-handed 2-spinor ρ and the equal clockwise rotation to λ, and the magnitude of j alone can be changed by boosting ρ forward and boosting λ backward.

Table 2: Changes $\{\Delta\theta, \Delta\phi, \Delta\alpha\}$ due to a rotation of angle κ about each coordinate axis. Values of $\{\theta, \phi, \alpha\}$ are provided that give the components of the eigenspinors. The x^1 and x^2 eigenspinors are not normalized.

Eigenspinor $\rightarrow$	x^1	x^1	x^2	x^2
Components $\rightarrow$	col$\{-1, 1\}$	col$\{1, 1\}$	col$\{i, 1\}$	col$\{-i, 1\}$
$\{\theta, \phi, \alpha\}$ $\rightarrow$	$\{\frac{\pi}{2}, \pi, -\pi\}$	$\{\frac{\pi}{2}, 0, 0\}$	$\{\frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}\}$	$\{\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}\}$
Rotation Axis $\downarrow$				
x^1-axis	$\{0, 0, +\kappa\}$	$\{0, 0, -\kappa\}$	$\{+\kappa, 0, 0\}$	$\{-\kappa, 0, 0\}$
x^2-axis	$\{-\kappa, 0, 0\}$	$\{+\kappa, 0, 0\}$	$\{0, 0, +\kappa\}$	$\{0, 0, -\kappa\}$
x^3-axis	$\{0, +\kappa, 0\}$	$\{0, +\kappa, 0\}$	$\{0, +\kappa, 0\}$	$\{0, +\kappa, 0\}$

Table 3: A continuation of Table 2

Eigenspinor $\rightarrow$	x^3	x^3
Components $\rightarrow$	col$\{0, 1\}$	col$\{1, 0\}$
$\{\theta, \phi, \alpha\}$ $\rightarrow$	$\{\pi, \phi_0, -\phi_0\}$	$\{0, \phi_0, \phi_0\}$
Rotation Axis $\downarrow$		
x^1-axis	$\{-\kappa, -\phi_0 + \frac{\pi}{2}, +\phi_0 - \frac{\pi}{2}\}$	$\{+\kappa, -\phi_0 - \frac{\pi}{2}, -\phi_0 - \frac{\pi}{2}\}$
x^2-axis	$\{-\kappa, -\phi_0 + \pi, +\phi_0 - \pi\}$	$\{+\kappa, -\phi_0, -\phi_0\}$
x^3-axis	$\{0, 0, +\kappa\}$	$\{0, 0, -\kappa\}$

A Problems

1. Find j, a, r, and s when
   (i) the 4-spinor ψ has four equal real-valued components: $A = a = c = e = g$ and $0 = b = d = f = h$;
   (ii) as in (i) but with c negative: $A = a = -c = e = g$ and $0 = b = d = f = h$;
   (iii) try $A = a = d = e = f$, $0 = b = c$, and $2A = g$.
2. Use the gammas (3) to find j as a function of a ... h.
3. Show that $\gamma^1 \cdot \gamma^2 + \gamma^2 \cdot \gamma^1 = 0$ and that $\gamma^2 \cdot \gamma^2 + \gamma^2 \cdot \gamma^2 = -2 \cdot 1$, where '1' is the unit 4x4 matrix.
4. By definition, $\exp[-i\sigma^1\kappa/2] \equiv \Sigma(-i\sigma^1\kappa/2)^n/n!$.
   (i) Calculate $(\sigma^1)^2 = \sigma^1 \cdot \sigma^1$.
   (ii) Show $\exp[-i\sigma^1\kappa/2] = \cos(\kappa/2) \cdot 1$ - $i \sin(\kappa/2)\sigma^1$, where '1' is the unit 2x2 matrix.
5. Find r, θ_R, ϕ_R, α, β, l, θ_L, and ϕ_L for the 4-spinor of problem 1(iii).
6. The parity operator P has the following effect on a 4-spinor in the chiral representation: $P\begin{pmatrix}\rho \\ \lambda\end{pmatrix} = \begin{pmatrix}-\lambda \\ -\rho\end{pmatrix}$, where ρ and λ are the right- and left-handed 2-spinors. The charge conjugation operator C has the following effect: $C\psi = i\gamma^2\psi$.
   Apply P, C and CP to the 4-spinor of problem 1(iii) and find the j's and a's.
7. (i) Find a 64 component quantity $\Gamma^\mu_{\nu\tau}$ so that $j^\mu = -\Gamma^\mu_{\nu\tau} r^\nu s^\tau$ and $\Gamma^\mu_{\nu\tau} = -\Gamma^\mu_{\tau\nu}$.
   (ii) Show that $0 = r^\mu + s^\mu + \Gamma^\mu_{\nu\tau} r^\nu s^\tau$. Interpret that equation using parallel transfer and the parallelograms of Figure 1.

References

[1] Among Quantum Mechanics books see, for example: Messiah, A., *Quantum Mechanics* (North Holland 1966), Volume 2, Chapter XX; Sakurai, J.J., *Advanced Quantum Mechanics* (Addison-Wesley 1967), Appendices B and C.

[2] Among Quantum Field Theory books see, for example: Itzykson, C. and Zuber, J., *Quantum Field Theory* (McGraw-Hill 1980), Appendix A-2; Berestetsky, V. B., Lifshitz, E. M., and Pitaevskii, L. P., *Quantum Electrodynamics* (Pergamon 1980), pp. 76-84; Weinberg, S., *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, 1995), Volume I, Section 5.4.